RESEARCH PAPER
Contrib. Sci. & Tech Eng, 2025, 2(4)
Homepage: https://cste.journals.umz.ac.ir/
DOI: 10.22080/cste.2025.29016.1030

# Stability of Nonlinear Fractional Order Discrete-Time Systems for Engineering Applications

**Amir Veisi [1], Hadi Delavri [2]***

[1] Department of Electrical Engineering, Faculty of Engineering, Bu-Ali Sina University, Hamedan, Iran

[2] Department of Electrical Engineering, Hamedan University of Technology, Hamedan, Iran

**Article Info**
Received 14 April 2025
Accepted 22 June 2025
Available online 14 September 2025

**Keywords:**
Stability Analysis;
Discrete Caputo Fractional Order Calculus;
Fractional Calculus;
Control Systems;
Lyapunov Stability.

**Abstract:**
Data-driven approaches, by effectively capturing complex nonlinear behaviors, have emerged as a powerful tool for enhancing control of real engineering systems. Nonlinear discrete-time fractional-order systems have earned much interest in the design of controllers and system modeling due to special characteristics that include long-term memory, an expansion of stability domains, and higher accuracy. It is thus essential to establish some straightforward theories for proving and analyzing the stability of these particular systems. This paper presents a sophisticated approach to the stability analysis of discrete Caputo fractional-order systems, developing a new Lyapunov-based stability analysis framework specifically tailored for these systems. The effectiveness of the proposed approach has been critically analyzed theoretically and validated through numerical simulations. Methodologically innovative, such reasoning thus provides a strong solution to the stability test problem of discrete fractional order systems, opening up greater avenues for the advancement of control theory and dynamical system theories within fractional calculus.



***Corresponding Author: delavari@hut.ac.ir**

**Supplementary information:** Supplementary information for this article is available at https://cste.journals.umz.ac.ir/

**Please cite this paper as:** Veisi, A. & Delavari, H. , (2025). Stability of Nonlinear Fractional Order Discrete-Time Systems for Engineering Applications. Contributions of Science and Technology for Engineering, 2(4), 1-12. doi:10.22080/cste.2025.29016.1030.

## 1. Introduction

The development of fractional calculus is a natural extension of the conventional notion of derivatives in ordinary differential Equations to those cases involving fractional-order derivatives. This evolution started in the 19th century, bringing about continuous questions as to how ordinary differential Equations could be extended to include fractional derivatives. Nowadays, the modern field has grown significantly, marked by numerous improvements and innovations [1]. The development of fractional calculus began in earnest in the 19th century with the introduction of the Riemann-Liouville approach, formulated by Bernhard Riemann and Joseph Liouville in the 1840s, laying the foundational principles. In 1867, Adolph Grunwald further developed the subject in what would later be called the Grünwald-Letnikov formulation [2]. The French mathematician Augustin-Louis Cauchy also contributed to this development with the introduction of the fractional integral. Cauchy, during the period from 1790 to 1857, discussed several mathematical problems, especially in mathematical analysis, and gave essential contributions to the theory of complex functions, including the concept of fractional integral [3]. In the twentieth century, the development of fractional calculus continued to evolve significantly. In 1934, Aleksei Letnikov completed Grunwald's work, bringing the Grünwald-Letnikov formulation to its final form. Later, in 1967, Michele Caputo developed another type of fractional calculus that was more applicable to physical and engineering applications, as it provided better modeling for initial conditions. In the last few decades of the twentieth century, broad research practice has been conducted in the realm of the fractional calculus, which extended and promoted the said calculus. Among them, the works of Augustin-Louis Cauchy [4], Yuri Luchko [5], and Anatoly Kilbas [6] are the most prominent and have really developed the study of fractional calculus. In the twenty-first century, fractional calculus has emerged as a rapidly growing field of research in mathematics and engineering. Current studies are being carried out on numerical methods, applications, and theoretical developments in fractional calculus. Besides, this field is also widely applied to the modeling and analysis of complex systems in many different fields, reflecting its versatility and importance. On that note, a historical view from the theoretical status of fractional calculus to a dynamic and operational research area is then recognized as a strong tool



ISSN 3060-6578

for modeling and analyzing complex systems [7]. Methods and systems exhibiting memory properties and non-Markovian behaviors can be represented and investigated only with the help of this significant and intriguing subfield of mathematics and engineering, known as fractional calculus. In this framework, three approaches typically represent the Fractional Calculus, one of which is the Grünwald-Letnikov (GL) approach. Other widely used definitions are those by Riemann-Liouville (RL) and Caputo.

### 1.1. Definition

In this section, attention will be given to the presentation of some definitions in fractional calculus and their inter-relations. One of the critical topics involving fractional calculus is indeed the fractional-order integral, which introduces the possibility of extending the notion of integration to fractional orders. In what follows, the fractional-order integral will be considered in the continuous and discrete cases. The continuous fractional integral generalizes the ordinary integral to perform integration to a fractional order, not necessarily an integer. The continuous fractional integral of order $\alpha$ is given, for instance, in a study by Yang and Zhang [8], by:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau \qquad (1)$$

where $I^{\alpha}$ denotes the fractional integral of order $\alpha$, $\Gamma(\alpha)$ is the gamma function, which generalizes the factorial function for real and complex numbers, $a$ is the lower limit of integration, $t$ is the upper limit of integration, and $\tau$ is the integration variable.

The discrete fractional integral generalizes the integration in discrete space applied to data and functions defined on integers. The discrete fractional integral of order $\alpha$ is given by Ferreira [9]:

$$\Delta^{-\alpha} f(k) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} (k-j+1)^{\alpha-1} f(j) \qquad (2)$$

where $\Delta^{-\alpha}$ represents the discrete fractional integral of order $\alpha$, $\Gamma(\alpha)$ is the gamma function, k is the upper limit of summation, and $j$ is the summation index. One of the advantageous sides of such a derivative consists in that its calculation is straightforward and effective, and is usually implemented numerically using such methods as Euler's method. However, its accuracy as an approximation to the real derivative is relatively low when functions with substantial deviations are considered. The GL fractional derivative of a function $f(t)$ reads as [8]:

$$\frac{d^{\alpha} f(t)}{dx^{\alpha}} = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t-kh) \qquad (3)$$

where $\alpha$ is a positive real number and $\binom{\alpha}{k}$ is the binomial coefficient defined as:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!} \qquad (4)$$

Another advantage of this class of derivatives is that they enjoy good properties in approximating functions such as analytic functions and functions containing singular points. However, this approach often requires complex and computationally intensive calculations, and sometimes necessitates the use of specialized functions and techniques. The Riemann-Liouville (RL) fractional derivative of a given function $f(t)$ can be expressed as:

$$D_{RL}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-t)^{n-\alpha-1} f(\tau) d\tau \qquad (5)$$

where $\alpha$ is a positive real number, $n = \lceil \alpha \rceil$, and $\Gamma(n)$ is the gamma function [8]. This Caputo definition is obtained through the modification of the Riemann-Liouville derivative. Compared with the RL derivative, it has an advantage—it satisfies certain initial conditions of fractional-order differential Equations, which might cause issues in the RL case. However, sometimes these derivatives encounter notable differences when applied to real or imaginary quantities. Defined in terms of left differential operators, the Caputo fractional-order derivative of a function $f(t)$ is expressed ~~as~~ by Yang and Zhang [8]:

$$D_C^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad n-1 < \alpha < n \qquad (6)$$

where $\alpha$ is a positive real number, $n = \lceil \alpha \rceil$, and $\Gamma(n)$ is the gamma function. For $n = 1$, Equation 6 will be defined as 7:

$$D_C^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau) d\tau \qquad (7)$$

where $0 < \alpha < 1$. Now, the discrete Caputo derivate Equation is defined [9, 10]:

$$\Delta_C^{\alpha} x(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha} x(k-j) \qquad (8)$$

The gamma function is crucial in this definition and is defined as follows [11]:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt \qquad (9)$$

This definition is valid for all real and complex numbers except non-positive integers. Here, the gamma function also plays a fundamental role. The discrete gamma function for a non-negative integer $n$ and a real number $\alpha$ is defined as follows [11]:

$$\Gamma(\alpha+n) = \prod_{k=0}^{n-1} (\alpha+k) = (\alpha)(\alpha+1)(\alpha+2)\ldots(\alpha+n-1) \qquad (10)$$

The Mittag-Leffler function, denoted by $E_{\alpha,\beta}(z)$ is a special function that generalizes the exponential function. It is defined as [12]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \qquad (11)$$

where $z$ is a complex number, $\alpha$ and $\beta$ are positive parameters, and $\Gamma(z)$ is the gamma function. In mathematics and physics, the Mittag-Leffler function is widely encountered in various contexts, including fractional calculus, rational differential Equations, and probability theory. It is also used for modeling multiple processes with memory effects, including viscoelastic materials, fractional-order systems, and anomalous diffusion. The Mittag-Leffler function exhibits interesting properties, such as being an entire function for all complex $\alpha$ and $\beta$, provided there is a branch cut along the negative real axis when $\alpha \geq 1$. Additionally, it possesses several integral representations and recurrence relations. This function plays a critical role in the analysis and solution of fractional differential and integral problems [12].

### 1.2. The Benefits of Using Discrete Caputo Fractional Calculus

Generally, the advantages of the discrete Caputo fractional derivative are classified into four classes. Definable Initial Conditions: One of the significant advantages of discrete Caputo fractional calculus is its ability to define initial conditions of fractional differential Equations easily. This property is beneficial in the study and solution of such Equations because it provides practical methods for solving complicated problems. Application to Dynamic System Analysis: The discrete Caputo fractional calculus is particularly well-suited for analyzing and modeling dynamic systems that exhibit memory properties and require precise initial conditions. Such an approach serves to help the engineers and researchers study the behavior of complex systems to analyze and optimize the processes of industry and engineering. Control Problems: The discrete Caputo fractional calculus also finds its application in control problems and systems of artificial intelligence. Considering the memory properties and non-Markovian behaviors in most of the applications of artificial intelligence and control challenges, the use of discrete Caputo fractional calculus can help in enhancing and optimizing the performance of such systems. Delay Differential Equations Modeling: Discrete Caputo fractional calculus can be used in modeling and analyzing delay differential Equations. This has a great value in investigations into the behavior of complex systems possessing memory and strange instabilities.

With its properties, discrete Caputo fractional calculus is an efficient and very useful tool in analyzing and modeling complex systems that might also be inhomogeneous. The methodology has thus been employed in improvements and optimizations of many industrial and engineering processes [13].

### 1.3. The Overview of Caputo, GL and RL

These definitions and theories, which have been reviewed for analysis and prediction, can be effectively applied to complex systems' behavior, such as dynamic systems and physical phenomena. All these methods have their specific characteristics, advantages, and limitations that need to be investigated and compared thoroughly. One of the most essential features of Caputo fractional calculus, in comparison with the other two definitions, GL and RL, is its ability to accurately define the initial conditions. Initial conditions can easily be established in Caputo fractional calculus, which, very often, simplifies the analysis of different problems. Caputo fractional calculus can model more realistic and complex issues effectively due to its features, such as the ability to define initial conditions and its relation with delay differential Equations. Another important strong point is that it provides a better approximation to real derivatives. Caputo fractional calculus generally yields higher precision in estimating real derivatives compared to other methods, such as GL and RL. Caputo Fractional Calculus typically provides higher accuracy in approximating real derivatives compared to its alternatives, such as GL and RL. Finally, due to the possibility of defining initial conditions and increased precision while approximating real derivatives, the Caputo fractional calculus can also find wide application in such directions as control problems, signal processing, and dynamic system analysis [14].

### 1.4. Applications of Caputo in Engineering

Dynamic Systems Modeling: Caputo fractional calculus has a wide range of applications in modeling dynamic systems with memory effects, such as viscoelastic materials [15], fluid flow in porous media [16], and electrical circuits with non-local effects [17]. Certainly, Caputo's fractional calculus is preferred for the accurate description of such systems because it may capture the memory and hereditary properties.

Control Systems: Applications, including the Caputo fractional calculus, can be found in designing controllers for control engineering when the dynamics of such complicated systems appear to be fractional. It provides a very successful modelling of these systems that belong to local effects. Due to this fact, it provides highly efficient control strategies, achieving robust performance in numerous industrial processes and systems.

Signal Processing: Caputo fractional calculus finds its application in signal processing for the analysis and processing of signals that have long-term memory or non-local dependencies. It provides tools for characterizing the behavior of signals in communication systems, biomedical signal analysis, image processing, and pattern recognition, among other areas [21].

Mechanical Engineering-Materials Science: Fractional calculus, as introduced by Caputo, is utilized in modeling the viscoelastic behavior of materials, such as polymers and biological tissues, which exhibit complex rheological properties, in the fields of mechanical engineering and materials science. This will enable engineers to predict the material response under various loading conditions, and therefore design structures for improved durability and performance [22].

Heat and Mass Transfer: Heat and mass transport phenomena in porous media, fractal geometries, composite materials, etc., have also been investigated with the Caputo fractional calculus. It enables modeling of non-Fickian diffusion and anomalous transport behavior, contributing to the optimization of heat exchangers, filtration systems, and other diffusion-controlled processes [23].

Renewable Energy Systems: Caputo fractional calculus in renewable energy engineering contributes to modeling and optimizing the performance of renewable energy systems, including photovoltaic panels, wind turbines, and energy storage devices. In this context, accounting for the non-local nature of energy conversion processes allows engineers to increase the efficiency and reliability of renewable energy technologies [24-29].

In general, Caputo fractional calculus has become an effective tool in engineering fields for modeling complex problems of memory, non-locality, and fractional dynamics, generating applications in the enhancement of system modeling and control, signal processing, materials science, and renewable energy.

### 1.5. A Comparison of Fractional Calculus

Finally, in this section, for the sake of clarity regarding the advantages and disadvantages of each method discussed, comparisons have been made across three perspectives: accuracy, initial conditions, and applications, with a comprehensive comparison [9].

**Accuracy**:

- **Caputo:** Caputo fractional calculus typically provides better approximations of real derivatives, especially for functions with significant deviations.
- **GL**: GL fractional calculus may have lower accuracy in approximating real derivatives, particularly for functions with sharp deviations.
- **RL:** RL fractional calculus also approximates real derivatives but may struggle to capture specific features accurately.

**Initial Conditions:**

- **Caputo:** Caputo fractional calculus naturally incorporates initial conditions, making it suitable for problems with well-defined initial values.
- **GL:** GL fractional calculus does not inherently include initial conditions and may require additional adjustments or techniques to handle them.
- **RL:** RL fractional calculus does not directly incorporate initial conditions, potentially complicating the analysis of specific problems.

**Applications:**

- **Caputo:** Caputo fractional calculus finds widespread use in engineering and scientific applications due to its versatility and ability to model various systems accurately.
- **GL:** GL fractional calculus is often preferred for its simplicity in numerical computations and is suitable for problems where initial conditions are not critical.
- **RL:** RL fractional calculus has theoretical advantages in specific applications, particularly in problems involving integral transforms, but may be less practical for numerical calculations.

In this article, the content is divided into four sections: (2) Review of necessary preliminaries, (3) Novel approach to Lyapunov stability analysis for discrete Caputo, (4) Examination of results and examples, and (5) Overall summary of the article.

## 2. Preliminaries

In this section, the introductory relations in the domains of continuous and discrete time will first be introduced.

**The Young Inequality Theory**

Young's inequality is introduced here and plays a significant role in proving power law inequalities [30].

**Lemma 1:** For any $a, b \geq 0$, if $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, Then:

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q \qquad (12)$$

where equality holds if and only if $a^p = b^q$.

## 3. Novel Approach to Lyapunov Stability Analysis for Discrete Caputo

Stability of dynamical systems is a very critical activity within the domain of applied mathematics and engineering. Among the standard approaches for system stability analysis, the Lyapunov theory has gained significant importance. In this paper, the stability of discrete Caputo fractional-order systems will be investigated based on Lyapunov stability theory. In the previous sections, some fundamental aspects of fractional-order dynamical systems have been introduced. This section tries to provide the proof of stability for such systems. Finally, based on Lyapunov theory, discrete Caputo systems are deeply analyzed, and stability is established.

### 3.1. Lyapunov Stability for Fractional-Order Systems

To examine the stability of discrete fractional-order dynamical systems, the Lyapunov function is commonly employed. Consider the following discrete fractional-order system given by Gajic Qureshi [31]:

$$\Delta^\alpha x[k] = f(x[k]), \;\; x[k] \in \mathbb{R}^n, \;\; 0 < \alpha \leq 1 \qquad (13)$$

where $\Delta^\alpha$ denotes the discrete Caputo fractional difference of order $\alpha$. A function $V: \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the given discrete fractional-order system if it satisfies the following conditions [31]:

❖ $V(x[k]) > 0$ for all $x[k] \neq 0$ and $V(0) = 0$.

❖ The forward difference of $V$ along the trajectories of the system, i.e., $\Delta V(x[k]) = V(x[k+1]) - V(x[k])$, is negative semi-definite.

**Theorem:**

Let $x(k) \in R$ be a real-valued discrete time function. $\mu = \frac{m}{n} \geq 1$, where $m > 0$ is an even number and $n \in N^+$, Then, for any discrete time instant $k \geq 1$, The following holds:

$$\Delta_C^\alpha\, x^\mu(k) \le \mu\, x^{\mu-1}(k)\, \Delta_C^\alpha\, x(k) \quad \forall\, \alpha(0,1) \tag{14}$$

**Proof:**

Now, the goal is to prove the following Equation:

$$\Delta_C^\alpha\, x^\mu(k) - \mu\, x^{\mu-1}(k)\, \Delta_C^\alpha\, x(k) \le 0 \quad \forall\, \alpha(0,1) \tag{15}$$

Let us define the auxiliary variable $\mathfrak{D}(k) = \Delta_C^\alpha\, x^\mu(k) - \mu\, x^{\mu-1}(k)\, \Delta_C^\alpha\, x(k)$. Now, having the discrete Caputo derivative relation in Equation 8, the relation can be obtained for $x^\mu(k)$.

$$\Delta_C^\alpha\, x^\mu(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha}\;\; x^\mu(k-j) \tag{16}$$

Similarly, it can be obtained that:

$$\mu\, x^{\mu-1}(k)\Delta_C^\alpha x(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha}\, \mu\, x^{\mu-1}(k)\, x(k-j) \tag{17}$$

By replacing the two above Equations in Equation 15, Equation 18 can be obtained:

$$\mathfrak{D}(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha}\;\; x^\mu(k-j) - \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha}\, \mu\, x^{\mu-1}(k)\, x(k-j) \le 0 \tag{18}$$

By factoring out standard terms from Equations 18, it will be simplified to Equation 19.

$$\mathfrak{D}(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha}\; [\, x^\mu(k-j) - \mu\, x^{\mu-1}(k)\, x(k-j)] \le 0 \tag{19}$$

Now, to further develop the simplification of Equation 19 and utilizing young inequality, it can be proven that:

$$x^{\mu-1}(k)\, x(k-j) \le |\, x^{\mu-1}(k)|.|\, x(k-j)| = a.b \tag{20}$$

By substituting Equation 20 into Equation 12, it will be obtained that:

$$\begin{aligned} ab &\le \frac{1}{p} a^p + \frac{1}{q} b^q \\ &= \frac{\mu-1}{\mu} |\, x^{\mu-1}(k)|^{\frac{\mu}{\mu-1}} + \frac{1}{\mu} |\, x(k-j)|^\mu \\ &= \frac{\mu-1}{\mu} \left|\, x^{\frac{m-n}{n}}(k)\right|^{\frac{m}{m-n}} + \frac{1}{\mu} |\, x(k-j)|^{\frac{m}{n}} \\ &= \frac{\mu-1}{\mu}\, x^\mu(k) + \frac{1}{\mu}\, x^\mu(k-j) \end{aligned} \tag{21}$$

With simplification you have that:

$$x^{\mu-1}(k)\, x(k-j) \le\; x^\mu(k) - \frac{1}{\mu}\, x^\mu(k) + \frac{1}{\mu}\, x^\mu(k-j) \tag{22}$$

In accordance with mathematical rules, it is possible to multiply both sides of Equation 22 by $\mu$.

$$\mu\, x^{\mu-1}(k)\, x(k-j) \le \mu\, x^\mu(k) -\; x^\mu(k) + x^\mu(k-j) \tag{23}$$

Then:

$$\mu\, x^{\mu-1}(k)\, x(k-j) \le (\mu-1)\, x^\mu(k) + x^\mu(k-j) \tag{24}$$

By replacing the above Equation in 19:

$$\mathfrak{D}(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha}\, (1-\mu)\, x^\mu(k) \le 0 \tag{25}$$

According to the rules of sigma notation, when is $a$ scalar, $\sum_j aj$ is equivalent to $a \sum_j j$.

$$\mathfrak{D}(k) = \frac{1}{\Gamma(1-\alpha)}\, (1-\mu)\, x^\mu(k) \sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha}\; \le 0 \tag{26}$$

For simplicity in determining the sign, the above relationship can be divided into two parts: $\frac{1}{\Gamma(1-\alpha)}\, (1-\mu)\, x^\mu(k)$, Due to the condition set for $\mu$, this semester has a negative sign. Now, if the following relation is positive, the general relation Equation 15 can be proved.

$$\sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha}\; \ge 0 \tag{27}$$

Due to the larger magnitude of $k$ compared to $j$, Equation (27) will always be non-negative. Thus, Equation 15 will always be less than or equal to zero. According to Lyapunov's stability theory, given that the candidate Lyapunov function is positive-definite and the fractional order discrete Caputo derivative of the candidate Lyapunov function is negative semi-definite, it will be proven that the system is stable in the Lyapunov sense. The image of Equation 27 is presented in Figure 1, illustrating the positive value of Equation 27.

As shown in the figure, with an increase in $\alpha$, the output approaches zero. However, for all values of $0 < \alpha < 1$, the result remains positive.

### 3.2. Properties of Discrete Derivate Caputo:

According to the essential properties of the fractional order Caputo derivative in the continuous domain, such as linearity and a constant number derivative, the aim here is to investigate these two necessary and practical characteristics of the Caputo derivative in the discrete domain. The linearity and constant derivative properties of the Caputo derivative offer several advantages, particularly in the context of solving differential Equations and modeling real-world phenomena.

**1- Linearity:** The Caputo derivative in the discrete domain obey linearity, meaning that for constant $\psi$ and $\delta$ and function $f(k)$ and $g(k)$, it satisfies:

$$\Delta_C^\alpha[\psi f(k) + \delta g(k)] = \psi \Delta_C^\alpha f(k) + \delta \Delta_C^\alpha g(k) \tag{28}$$

On other hand, we know $\Delta x(k) = x(k) - x(k-1)$. By substitution and simplification, Equation 28 will be obtained.

$$\Delta_C^\alpha[\psi f(k) + \delta g(k)] = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha}\, \Delta\big(\psi f(k) + \delta g(k)\big) = \frac{\psi}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha}\, \Delta f(k) + \frac{\delta}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha}\, \Delta g(k) = \psi \Delta_C^\alpha f(k) + \delta \Delta_C^\alpha g(k) \quad (29)$$

The linearity property of the discrete Caputo, as stated in Equation 28, has been proven. The linearity property of the discrete Caputo fractional calculus has led to unique characteristics in the discretized Caputo formulation, some of which are listed below [9]:

**Simplification of Calculations:** Linearity allows for the superposition principle, meaning that the derivative of a sum of functions is the sum of their derivatives. This property simplifies the process of solving differential Equations because it allows for the breaking down of complex problems into simpler parts.

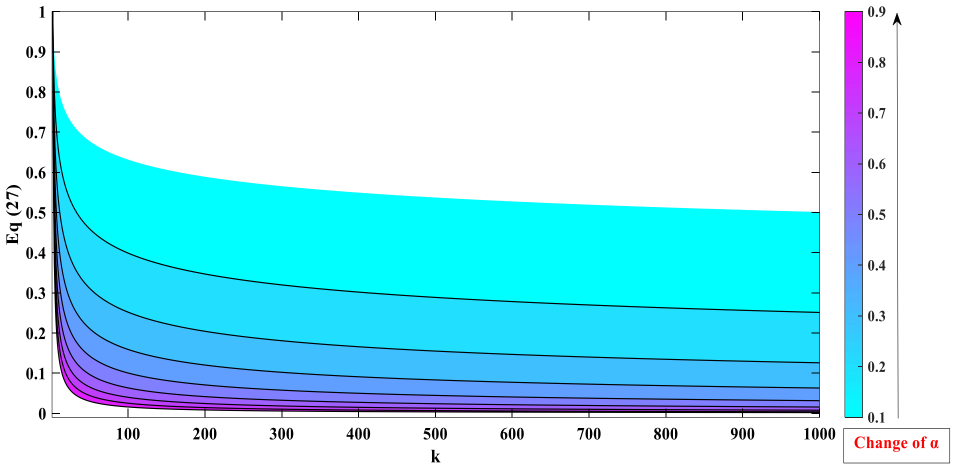


**Figure 1. The output of Equation 27**

**Analytical Solutions:** Many physical systems and processes can be described as linear combinations of simpler functions. The linearity of the Caputo derivative helps in finding analytical solutions to these problems.

**2- Derivative of a Constant:** For a constant function $C$, the discrete fractional derivative is zero: $\Delta_C^\alpha[C] = 0$.

$$\Delta_C^\alpha[C] = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1+\alpha} (k-j-1)^{-\alpha}\, \Delta C = 0 \quad (30)$$

With respect to $\Delta C = 0$, it is proven that the constant number derivative of discrete Caputo is equal to zero, similar to the integer order derivative and continuous Caputo. The advantage of the constant derivative in the Caputo derivative for engineering has several aspects:

**Accurate Modeling:** In many engineering problems, the initial conditions of systems play a crucial role in analysis and design. The Caputo derivative, with its transparent and independent initial conditions from higher-order derivatives, allows for more accurate modeling of system behaviors.

**Determining Equilibrium Points:** In the stability analysis of engineering systems, constants represent equilibrium points. The fact that the derivative of a constant in Caputo is zero facilitates a more straightforward and accurate stability analysis of these points.

**Predicting Long-term Behavior**: Given this property, engineers can better predict the long-term behavior of systems and design more robust solutions.

**Connection with Classical Differential Equations:** Many engineers are familiar with classical differential calculus. The property of the constant derivative in Caputo, which is similar to classical differential calculus, makes the transition to using fractional models more straightforward and more understandable.

The Caputo derivative finds significant application in modeling complex materials and systems. One area of application lies in modeling viscoelastic materials, which exhibit properties that depend on their stress and strain history. The constant derivative and linearity of the Caputo derivative facilitate more accurate modeling and improved simulation of the behaviors of such materials. Additionally, in the realm of complex dynamic systems characterized by nonlinear behaviors, the utilization of the Caputo derivative provides engineers with enhanced analytical and design tools. By leveraging its inherent properties, the Caputo derivative enables a deeper understanding and more effective manipulation of these systems, contributing to advancements in various engineering domains.

### 3.3. The Specific Features of This Article

The primary objective of this article is to propose a general approach for proving stability in both integer-order and fractional-order discrete Caputo systems. This implies that the established stability can be utilized for all $\mu$. The approach is based on the practicality of the theory presented in modeling and particularly in control engineering design. The presented method ensures that discrete Caputo fractional calculus can be employed in practical engineering systems with any degree of freedom. The article's approach establishes this assurance through a simple lemma (Yang's lemma), making the proof of discrete Caputo usage very straightforward and among the shortest engineering paths, without needing deep and complex mathematical definitions, theories, or lemmas. One of the most essential features of this article is its simplicity in design and applicability in both integer and fractional-order systems, which will enhance flexibility in designing various controllers.

## 4. Example and Result

In this section, the aim is to prove and demonstrate the claims made in previous sections. In this regard, studies are divided into two parts: theoretical reviews and simulation studies (in MATLAB). Initially, the focus will be on examining three theoretical examples of stability proofs. The use of discrete-time Caputo fractional calculus has been extensively studied and applied in various references for modeling and analysis of systems [32-34].

### 4.1. Theoretical examples

The aim here is to analyze and verify the stability of a discrete-time fractional-order system. Where three examples will be examined.

**Example 1:** Take the following nonlinear discrete system model into consideration:

$$\Delta_C^{\alpha}|x(k+1)| = -f(k) \tag{31}$$

where $f(k) = |sin\ (k)|$. The Lyapunov candidate function is defined with a specific condition of positivity as follows $V(k) = |x(k+1)|$. Taking the fractional discrete Caputo derivative of both sides of the Lyapunov candidate function Equation 32 yields:

$$\Delta_C^{\alpha}V(k) \leq \Delta_C^{\alpha}|x(k+1)| \tag{32}$$

Substituting $f(k)$ into Equation 31 and then substituting the result into Equation 32, the following expression is derived:

$$\Delta_C^{\alpha}V(k) \leq -|sin\ (k)| \leq 0 \tag{33}$$

Negativity of the derivative of the Lyapunov candidate function implies stability in the Lyapunov sense.

**Remark 1:** Equation 14 is considered for the most common case where $\mu = 2$, which can be highly significant in proving the stability of control systems.

$$\Delta_C^{\alpha}\, x^2(k) \leq 2x(k)\, \Delta_C^{\alpha}\, x(k) \quad \forall\, \alpha(0,1) \tag{34}$$

$\mu = 2$ is a particular case of Equation 14, which can be used as a predefined positive function to prove stability in the Lyapunov sense.

**Example 2:** Examine the nonlinear system model:

$$\begin{aligned} &\Delta_C^{\alpha}x_1(k) = -sin^2(k)x_1(k) - sin(k)\,cos(k)x_2(k) \\ &\Delta_C^{\alpha}x_2(k) = -\,sin(k)\,cos(k)x_1(k) - cos^2(k)x_2(k) \end{aligned} \tag{35}$$

The Lyapunov function is defined as Equation 36, subject to a positivity condition.

$$V\big(x_1(k), x_2(k)\big) = \tfrac{1}{2}x_1^2(k) + \tfrac{1}{2}x_2^2(k) \tag{36}$$

By taking the discrete Caputo fractional derivative of both sides of Equation 36 and substituting the proven relationship Equation 34, Equation 37 will be defined.

$$\Delta_C^{\alpha}V\big(x_1(k), x_2(k)\big) \leq\ x_1(k)\Delta_C^{\alpha}x_1(k) + x_2(k)\Delta_C^{\alpha}x_2(k) \tag{37}$$

Now, by substituting Equation 35 into Equation 37, Equation 38 is derived.

$$\Delta_C^{\alpha}V\big(x_1(k), x_2(k)\big) \leq\ x_1(k)\big(-sin^2(k)x_1(k) - sin(k)\,cos(k)x_2(k)\big) + x_2(k)\big(-\,sin(k)\,cos(k)x_1(k) - cos^2(k)x_2(k)\big) \tag{38}$$

Simplification yields the computation as follows:

$$\Delta_C^{\alpha}V\big(x_1(k), x_2(k)\big) \leq\ -\big(sin(k)\,x_1(k) + cos(k)x_2(k)\big)^2 \leq 0 \tag{39}$$

According to Equation 39, the system is stable.

**Example 3:** Consider the nonlinear discrete system model:

$$\begin{aligned} &X(k+1) = Ax(k) + Bu(k) \\ &y(k) = Cx(k) + Du(k) \end{aligned} \tag{40}$$

In the state-space model written in 40, A and B are nonlinear functions. In this example, the objective is initially to design a fractional-order Caputo controller and then prove the stability of the control law. The error signal and the sliding surface are defined in Equation 41.

$$\begin{aligned} &e(k+1) = x(k+1) + x_d(k+1) \\ &s(k) = \lambda e(k) + \Delta_C^{-\alpha}e(k+1) \end{aligned} \tag{41}$$

In this context, the objective is to design a controller based on fractional calculus and prove the stability of the control law. Next, the fractional discrete Caputo derivative is applied to both sides of the sliding surface.

$$\Delta_C^{\alpha}s(k) = \lambda\Delta_C^{\alpha}e(k) + e(k+1) = -ksign\big(s(k)\big) \tag{42}$$

On one hand, one of the properties of the discrete and continuous Caputo derivative is equal to $\Delta_C^{\alpha}\big(\Delta_C^{-\alpha}f(k)\big) = f(k)$. By substituting Equation 41 into Equation 42 and performing the necessary simplifications, Equation 43 will be defined.

$$u(k) = -B^{-1}\left(\lambda \Delta_C^\alpha e(k) + ksign\big(s(k)\big) + Ax(k) - x_d(k+1)\right) \quad (43)$$

The stability of the specified sliding surface is analyzed in the subsequent step. A Lyapunov function, satisfying the positivity condition, is introduced and expressed in Equation 44.

$$V(k) = \frac{1}{2} s^2(k) \quad (44)$$

By taking the discrete Caputo fractional derivative of both sides of Equation 44 and substituting the proven relationship Equation 34, Equation 45 will be defined.

$$\Delta_C^\alpha V(k) \leq s(k)\Delta_C^\alpha s(k) \quad (45)$$

Now, by substituting Equation 42 into Equation 45, Equation 46 is obtained.

$$\begin{aligned} \Delta_C^\alpha V(k) &\leq s(k)\left(-ksign\big(s(k)\big)\right) \\ \Delta_C^\alpha V(k) &\leq -k|s(k)| \end{aligned} \quad (46)$$

It is stable in the Lyapunov sense.

**Remark 2:** One of the most significant challenges in modeling and design is ensuring the stability of the designed system. Therefore, one of the well-known theories for proving stability in nonlinear systems is the Lyapunov theory. As theoretically demonstrated, the proven relationship in Equation 14 can easily assist in proving Lyapunov stability. This allows for the use of the discrete Caputo fractional calculus in system design and modeling, controller design, etc., besides proving its stability. The proof provided in this paper will assure researchers that they can use this discrete Caputo calculus to benefit from its numerous advantages, which have been extensively discussed in the previous sections of the paper regarding its properties and wide-ranging applications.

**Remark 3:** To examine the claim stated in the article, three different theoretical examples have been studied: **Example 1:** Proof of the approach proposed for a discrete-time fractional-order system (applicable to Markov processes). **Example 2:** Proof of the approach proposed for a discrete fractional-order system (applicable to time stepping). **Example 3:** Examination of the approach proposed for a discrete-time integer-order state space model (relevant to Markov processes). Particularly in Example 3, which presents a comprehensive form of the system model, the practical applicability of the proposed approach in controller design is evident. This leads to operational freedom, flexibility, and increased stability in the region, etc., which have been fully addressed in previous sections.

## 4.2. Numerical Examples

In the previous sections of this article, the theoretical aspects of Equation 14 and its properties were proven and its applications were examined. In this section, the aim is to investigate the proven Equation through simulation in the MATLAB software environment, ensuring that Equation 26 consistently holds true.

**Example 1:** Consider Equation 26 $\left(\mathfrak{D}(k) = \frac{1}{\Gamma(1-\alpha)}(1-\mu)\, x^\mu(k) \sum_{j=0}^{k-1+\alpha}(k-j-1)^{-\alpha} \leq 0\right)$ for the widely used special case of $\mu = 2$ and the nonlinear function Equation 47, which will result in Equation 48.

$$x(k+1) = exp^{(tanh(k)+sin(2*pi))} \quad (47)$$

$$\mathfrak{D}(k) = \frac{-1}{\Gamma(1-\alpha)}\left(exp^{(tanh(k)+sin(2*pi))}\right)^2 \sum_{j=0}^{k-1+\alpha}(k-j-1)^{-\alpha} \leq 0 \quad (48)$$

Equation 48 was simulated for $k = 1{:}1000$, $\alpha = 0.1{:}0.1{:}0.9$, and the result is shown in Figure 2.

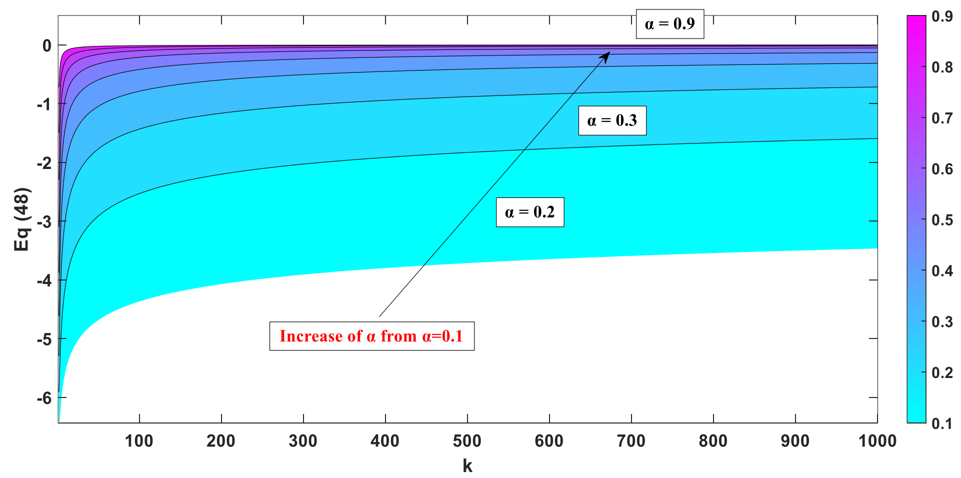


**Figure 2. Changes in $\alpha$ in Discrete Caputo Fractional Calculus**

As shown in Figure 2, the reduction in $\alpha$ leads to an increase in the stability region, which is also very similar to the results obtained using fractional calculus (when $0 < \alpha < 1$). The stability surface for the function introduced in Figure 3 is depicted in three dimensions, indicating an increase in the stability region with decreasing $\alpha$.

**Remark 4:** In the previous sections, after proving relation Equation 14, three theoretical examples and one simulated example were presented, all demonstrating the validity and applicability of the relation introduced in the paper. In this example, after examining the theoretical proof of stability, the aim is to illustrate the discrete Caputo fractional derivative presented in the paper.

**Example 2:** This example is significant in demonstrating the application of the presented theory in proving the Lyapunov stability for discrete nonlinear systems and models. A discrete nonlinear model is considered in Equation 31. Where $k = 1:3:300$. Where $f(k) = |\Delta_C^{\alpha}(si\,n(k))|$. The Lyapunov candidate function is defined with a specific condition of positivity as follows $V(k) = |x(k+1)|$. Taking the fractional discrete Caputo derivative of both sides of the Lyapunov candidate function Equation 49 yields:

$$\Delta_C^{\alpha} V(k) \leq \Delta_C^{\alpha}|x(k+1)| = -|\Delta_C^{\alpha}(sin\,(k))| \leq 0 \tag{49}$$

Negativity of the derivative of the Lyapunov candidate function implies stability in the Lyapunov sense. Here, the goal is to demonstrate $\Delta_C^{\alpha} V(k) \leq 0$ in the simulation, which represents the time plot of $\Delta_C^{\alpha} V(k)$, the results of which are shown in Figure 4.

As indicated in Figure 4, the stability level increases with a decrease in $\alpha$.

**Example 3:** Consider Equation 26 $\left(\beta(k) = \frac{1}{\Gamma(1-\alpha)}(1-\mu)\, x^{\mu}(k) \sum_{j=0}^{k-1+\alpha}(k-j-1)^{-\alpha} \leq 0\right)$ for the widely used special case of $\mu = 4$ and the nonlinear function Equation 47, which will result in Figure 5.

**Example 4:** In this section, the fractional-order mass-spring-damper system model is defined as $\left(F = m\,\Delta_C^2\, X(k) + c\,\Delta_C^{\alpha}\, X(k) + k\, X(k)\right)$. The model is simulated for a unit step input, and its stability region for $\alpha = 0.9$ is depicted in Figure 6. Where $m, c, k = 1, 0.4, 2$ are mass, damping coefficient, and spring constant, respectively.

**Stability:** The Lyapunov candidate function is defined with a specific condition of positivity as follows $V(k) = 0.5\, X(k)^2$. Taking the fractional discrete Caputo derivative of both sides of the Lyapunov candidate function Equation 50 yields:

$$\Delta_C^{\alpha} V(k) \leq X(k)\, \Delta_C^{\alpha}\, X(k) \leq 0 \tag{50}$$

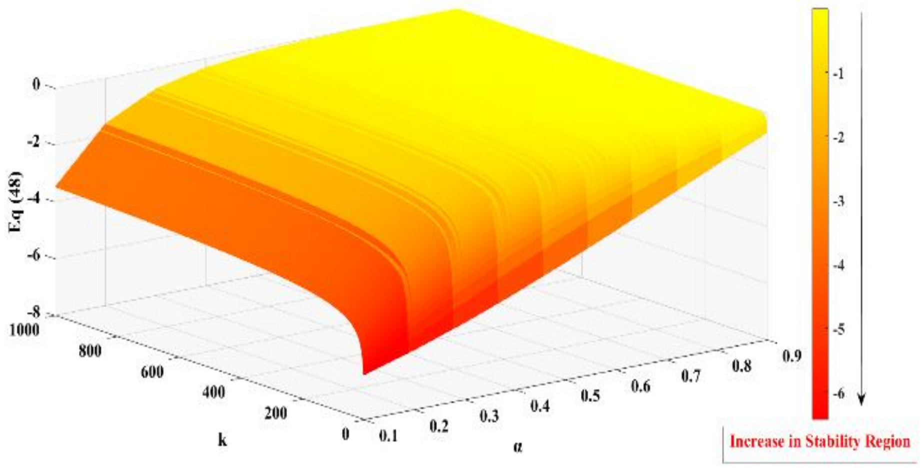


**Figure 3.** **Three dimensions in Equation 48 $(\mathfrak{D}(k))$**

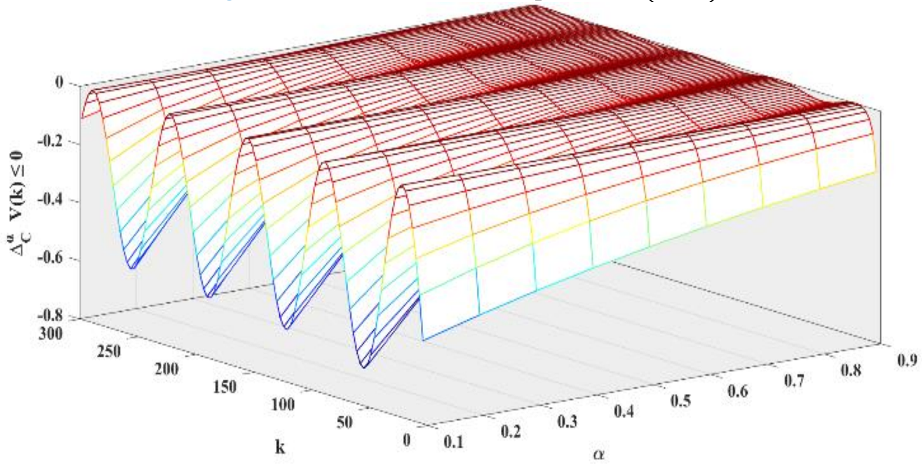


**Figure 4.** **The time plot of $\Delta_C^{\alpha} V(k)$**

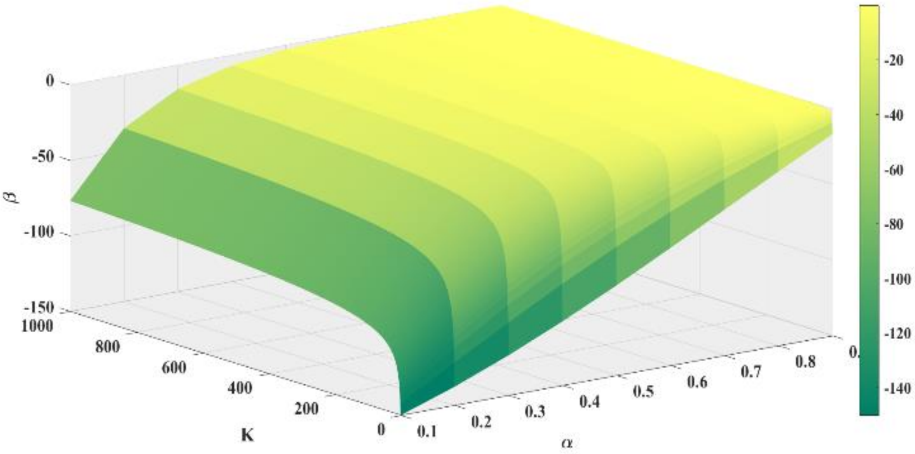


**Figure 5.** **Three dimensions in $\beta(k)$**

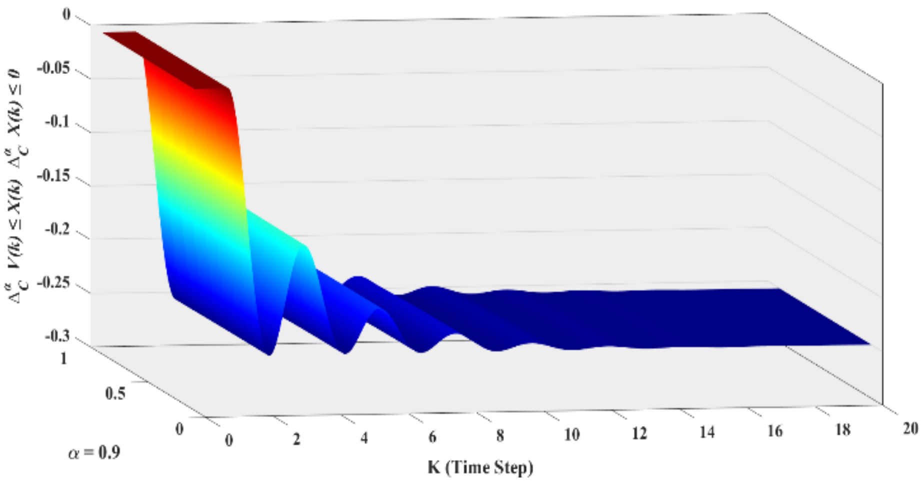


**Figure 6.** **The time plot of $X(k)$ $\Delta_C^\alpha$ $X(k)$**

**Remark 5:** The mass-spring-damper model serves as an analytical and simulation tool in various engineering fields. For instance, Chaotic Systems: When specific conditions are met for a mass-spring-damper system, it can transform into a chaotic system where small inputs lead to complex and unpredictable outputs. Vibratory Systems: This model can be applied to analyze vibratory systems, such as standard systems like fans or mechanical machines with oscillatory motions. Thermal-Energy Systems: In some cases, this model can be compared with systems related to heat transfer and thermal energy, such as thermal models that simulate temperature changes and heat transfer. Control Systems: In control engineering, the mass-spring-damper model is used to analyze the response of control systems to external inputs and noise, especially in studying dynamic response and robustness. Automotive Systems: In the design and optimization of automotive suspension systems, including travel and suspension systems, the mass-spring-damper model is utilized for simulating and optimizing vibrations. Mechanical and Industrial Systems: In the analysis and design of mechanical equipment such as industrial steering and suspension systems, this model is used to study dynamic response and vibrations. Energy Systems: In the study and optimization of energy generation systems, such as turbines, the mass-spring-damper model is used to model vibrations and dynamic responses. Electronic Systems: In the design and development of electronic systems prone to mechanical vibrations, this model is used to investigate the effects of vibrations on system performance. Structural Systems: In the analysis and optimization of structures and building systems, such as bridges and high-rise buildings, this model is used to simulate the dynamic behavior and vibrations of structures.

These explanations illustrate that the mass-spring-damper model is a powerful tool extensively used in the analysis and optimization of various engineering systems, highlighting its significant importance in the design and optimization of complex systems.

**Remark 6:** In order to substantiate the claims made, two simulation examples have been presented in this section. In Examples 1 and 2, the effects of variations in the fractional-order Caputo derivative for discrete-time systems (applicable to Markov processes) of both integer and fractional orders are clearly demonstrated in the figures above. Example 3 shows the validity of the proven relationship for various $\mu$, which can be particularly applicable in high-degree engineering systems. The results indicate the practical applicability of this study for discrete integer-order and fractional-order systems for different $\mu$, representing the degree of the system model in engineering. In Example 4 to demonstrate the relevance of the discrete Caputo fractional-order approach in proving stability for engineering systems, this example utilizes a mass-spring-damper system, and its stability region is shown in the figure.

## 5. Conclusion

A tailored discrete Lyapunov stability approach was employed in this work for the first time to analyze the stability of Caputo fractional-order systems. To achieve this objective in developing a specialized approach to that class of systems, effectiveness is demonstrated from both theoretical and numerical perspectives. Indeed, simulation results are provided herein and demonstrate theoretically and numerically how this method can constitute a strong tool within a wide range of applications concerning the problem at hand, specifically in the setting of discrete fractional-order systems analysis and stability. This method has great potential to contribute to the development of control theory and dynamical systems in the context of fractional calculus. Therefore, it is believed that this research can pave the way for further studies and practical applications in related fields.

## 6. References

[1] Das, S. (2011). Functional fractional calculus. Berlin: Springer. doi:10.1007/978-3-642-20545-3.

[2] Xue, D., & Bai, L. (2024). Fractional Calculus, High-Precision Algorithms and Numerical Implementations. Springer Singapore. doi:10.1007/978-981-99-2070-9.

[3] Zayed, A. I. (2024). Fractional integral transforms: Theory and applications. Chapman and Hall/CRC, Boca Raton, United States. doi:10.1201/9781003089353.

[4] Belhoste, B. (2012). Augustin-Louis Cauchy: A Biography. Springer, Cham, Switzerland.

[5] Al-Refai, M., & Luchko, Y. (2023). General Fractional Calculus Operators of Distributed Order. Axioms, 12(12), 1075. doi:10.3390/axioms12121075.

[6] Gorenflo, R., Kilbas, A. A., Mainardi, F., & Rogosin, S. (2020). Mittag-Leffler Functions, Related Topics and Applications. Springer Monographs in Mathematics. Springer, Berlin, Germany. doi:10.1007/978-3-662-61550-8.

[7] Singh, H., Srivastava, H. M., & Pandey, R. K. (2023). Special Functions in Fractional Calculus and Engineering. CRC Press, Boca Raton, United States. doi:10.1201/9781003368069.

[8] Yang, Y., & Zhang, H. H. (2019). Fractional calculus with its applications in engineering and technology. Synthesis Lectures on Mechanical Engineering, Springer, Cham, Switzerland, doi:10.1007/978-3-031-79625-8 .

[9] Ferreira, R. A. (2022). Discrete fractional calculus and fractional difference equations. SpringerBriefs in Mathematics, Springer, Cham, Switzerland. doi:10.1007/978-3-030-92724-0 .

[10] Abdeljawad, T. (2011). On Riemann and Caputo fractional differences. Computers and Mathematics with Applications, 62(3), 1602–1611. doi:10.1016/j.camwa.2011.03.036.

[11] Sebah, P., & Gourdon, X. (2002). Introduction to the gamma function. American Journal of Scientific Research, 2.

[12] Yuldashova, H., & Hasanov, A. (2024). Mittag-Leffler type functions of three variables. doi:10.22541/au.171604045.56900857/v1.

[13] Ostalczyk, P. (2015). Discrete Fractional Calculus: Applications in Control and Image Processing. World Scientific Publishing Company, Singapore. doi:10.1142/9833.

[14] Lubich, Ch. (1986). Discretized Fractional Calculus. SIAM Journal on Mathematical Analysis, 17(3), 704–719. doi:10.1137/0517050.

[15] Bhangale, N., Kachhia, K. B., & Gómez-Aguilar, J. F. (2023). Fractional viscoelastic models with Caputo generalized fractional derivative. Mathematical Methods in the Applied Sciences, 46(7), 7835–7846. doi:10.1002/mma.7229.

[16] Haq, S. U., Khan, M. A., Khan, Z. A., & Ali, F. (2020). MHD effects on the channel flow of a fractional viscous fluid through a porous medium: An application of the Caputo-Fabrizio time-fractional derivative. Chinese Journal of Physics, 65, 14–23. doi:10.1016/j.cjph.2020.02.014.

[17] Abro, K. A., Memon, A. A., & Uqaili, M. A. (2018). A comparative mathematical analysis of RL and RC electrical circuits via Atangana-Baleanu and Caputo-Fabrizio fractional derivatives. European Physical Journal Plus, 133(3), 1–9. doi:10.1140/epjp/i2018-11953-8.

[18] Veisi, A., & Delavari, H. (2022). Fractional-order backstepping strategy for fractional-order model of COVID-19 outbreak. Mathematical Methods in the Applied Sciences, 45(7), 3479–3496. doi:10.1002/mma.7994.

[19] Veisi, A., Maleki, H., & Delavari, H. (2023). Adaptive Fractional Sliding Mode Controller for Controlling Airway Pressure in an Artificial Ventilation System. 2023 9th International Conference on Control, Instrumentation and Automation, ICCIA 2023. doi:10.1109/ICCIA61416.2023.10506394.

[20] Delavari, H., & Veisi, A. (2023). Fuzzy fractional-order sliding mode control of COVID-19 virus variants. Computational Intelligence in Electrical Engineering, 14(1), 93-108.

[21] Ortigueira, M. D. (2024). Principles of fractional signal processing. Digital Signal Processing, 149, 104490. doi:10.1016/j.dsp.2024.104490.

[22] Cattani, C., Srivastava, H. M., & Yang, X.-J. (Eds.). (2015). Fractional Dynamics. Walter de Gruyter, Berlin, Germany. doi:10.1515/9783110472097.

[23] Imran, M. A., Khan, I., Ahmad, M., Shah, N. A., & Nazar, M. (2017). Heat and mass transport of differential type fluid with non-integer order time-fractional Caputo derivatives.

Journal of Molecular Liquids, 229, 67–75. doi:10.1016/j.molliq.2016.11.095.

[24] Delavari, H., & Veisi, A. (2021). Robust Control of a Permanent Magnet Synchronous Generators based Wind Energy Conversion. 2021 7th International Conference on Control, Instrumentation and Automation, ICCIA 2021. doi:10.1109/ICCIA52082.2021.9403590.

[25] Veisi, A., & Delavari, H. (2021). Adaptive Fractional order Control of Photovoltaic Power Generation System with Disturbance Observer. 2021 7th International Conference on Control, Instrumentation and Automation, ICCIA 2021. doi:10.1109/ICCIA52082.2021.9403598.

[26] Delavari, H., & Veisi, A. (2024). A new robust nonlinear controller for fractional model of wind turbine based DFIG with a novel disturbance observer. Energy Systems, 15(2), 827–861. doi:10.1007/s12667-023-00566-3.

[27] Veisi, A., & Delavari, H. (2024). Adaptive optimized fractional order control of doubly-fed induction generator (DFIG) based wind turbine using disturbance observer. Environmental Progress & Sustainable Energy, 43(2), 14087. doi:10.1002/ep.14087.

[28] Veisi, A., Delavari, H., & Shanaghi, F. (2023). Maximum Power Point Tracking in a Photovoltaic System by Optimized Fractional Nonlinear Controller. 2023 8th International Conference on Technology and Energy Management, ICTEM 2023. doi:10.1109/ICTEM56862.2023.10083639.

[29] Veisi, A., & Delavari, H. (2023). Adaptive fractional backstepping intelligent controller for maximum power extraction of a wind turbine system. Journal of Renewable and Sustainable Energy, 15(6). doi:10.1063/5.0161571.

[30] Chen, G. S. (2011). A generalized Young inequality and some new results on fractal space. arXiv preprint arXiv:1107.5222.

[31] Gajic, Z., & Qureshi, M. T. J. (2008). Lyapunov matrix equation in system stability and control. Courier Corporatio, Chelmsford, United States.

[32] Veisi, A., & Delavari, H. (2025). Deep reinforcement learning optimizer based novel Caputo fractional order sliding mode data driven controller. Engineering Applications of Artificial Intelligence, 140, 109725. doi:10.1016/j.engappai.2024.1097255.

[33] Veisi, A., & Delavari, H. (2024). Fractional data driven controller based on adaptive neural network optimizer. Expert Systems with Applications, 257. doi:10.1016/j.eswa.2024.125077.

[34] Fu, H., & Kao, Y. (2025). Robust sliding mode control of discrete fractional difference chaotic system. Nonlinear Dynamics, 113(2), 1419–1431. doi:10.1007/s11071-024-10279-6.